EXISTENCE FOR A CLASS OF DISCRETE HYPERBOLIC PROBLEMS

RODICA LUCA

Received 1 November 2005; Revised 10 March 2006; Accepted 7 April 2006

We investigate the existence and uniqueness of solutions to a class of discrete hyperbolic systems with some nonlinear extreme conditions and initial data, in a real Hilbert space.

1. Introduction

Let H be a real Hilbert space with the scalar product $\langle \cdot, \cdot \rangle$ and the associated norm $\|\cdot\|$. In this paper we will investigate the discrete hyperbolic system

$$\left.\begin{aligned} &\frac{du_j}{dt}(t)+\frac{v_j(t)-v_{j-1}(t)}{h_j}+A(u_j(t))\ni f_j(t),\\ &\frac{dv_j}{dt}(t)+\frac{u_{j+1}(t)-u_j(t)}{h_j}+B(v_j(t))\ni g_j(t),\end{aligned}\right.\quad j=\overline{1,N},\ t\in[0,T],\ \text{in } H, \tag{$\overline{S}$}$$

with the extreme conditions

$$v_0(t)=-\alpha(u_1(t)),\quad u_{N+1}(t)=\beta(v_N(t)),\quad t\in[0,T], \tag{$\overline{EC}$}$$

and the initial data

$$u_j(0)=u_{j0},\quad v_j(0)=v_{j0},\quad j=\overline{1,N}, \tag{$\overline{ID}$}$$

where $N\in\mathbb{N}$, $h_j>0$, $j=\overline{1,N}$, and α, β, A, B are operators in H, which satisfy some assumptions.

Hindawi Publishing Corporation
Advances in Difference Equations
Volume 2006, Article ID 89260, Pages 1–14
DOI 10.1155/ADE/2006/89260

This problem is a discrete version with respect to x (with $H = \mathbb{R}$) of the problem

$$\begin{aligned} &\frac{\partial u}{\partial t}(t,x) + \frac{\partial v}{\partial x}(t,x) + A(u(t,x)) \ni f(t,x), \\ &\frac{\partial v}{\partial t}(t,x) + \frac{\partial u}{\partial x}(t,x) + B(v(t,x)) \ni g(t,x), \end{aligned} \quad 0 < x < 1,\ t > 0, \text{ in } \mathbb{R}, \tag{S}$$

with the boundary conditions

$$v(t,0) = -\overline{\alpha}(u(t,0)), \quad u(t,1) = \overline{\beta}(v(t,1)), \quad t > 0, \tag{BC}$$

and the initial data

$$u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad 0 < x < 1. \tag{IC}$$

The above problem has applications in electrotechnics (the propagation phenomena in electrical networks) and mechanics (the variable flow of a fluid)—see [7, 8, 13]. The system (S) subject to various boundary conditions has been studied by many authors: Barbu, Iftimie, Moroşanu, and Luca, in [4, 5, 9, 11, 12]. Using an idea from [14] we discretize the problem $(S)+(BC)+(IC)$ in this way: let N be a given integer $(N \geq 1)$ and $h = 1/(N+1)$. In a first stage we approximate the system (S) and the boundary conditions (BC) by

$$\begin{aligned} &\frac{\partial u}{\partial t}(t,x) + \frac{v(t,x) - v(t,x-h)}{h} + A(u(t,x)) \ni f(t,x), \quad x \in (h,(N+1)h), \\ &\frac{\partial v}{\partial t}(t,x) + \frac{u(t,x+h) - u(t,x)}{h} + B(v(t,x)) \ni g(t,x), \quad x \in (0,Nh),\ t > 0, \\ &\qquad v(t,0) = -\overline{\alpha}(u(t,0)), \qquad u(t,Nh) = \overline{\beta}(v(t,Nh)). \end{aligned} \tag{1.1}$$

We look for u and v of the form $u(t,x) = \sum_{j=0}^{N} u_j(t)\varphi_j(x)$ and $v(t,x) = \sum_{j=0}^{N} v_j(t)\varphi_j(x)$, where

$$\varphi_j(x) = \chi_{[jh,(j+1)h)}(x) = \begin{cases} 1, & x \in [jh,(j+1)h), \\ 0, & x \notin [jh,(j+1)h). \end{cases} \tag{1.2}$$

We write f, g, u_0, v_0 as

$$\begin{aligned} f(t,x) &= \sum_{j=0}^{N} f_j(t)\varphi_j(x), \qquad g(t,x) = \sum_{j=0}^{N} g_j(t)\varphi_j(x), \\ u_0(x) &= \sum_{j=0}^{N} u_{j0}\varphi_j(x), \qquad v_0(x) = \sum_{j=0}^{N} v_{j0}\varphi_j(x), \end{aligned} \tag{1.3}$$

where $f_j(t) = f(t,jh)$, $g_j(t) = g(t,jh)$, $u_{j0} = u_0(jh)$, and $v_{j0} = v_0(jh)$.

Then for u_j and v_j we obtain the system

$$\begin{aligned} u_j' + \frac{v_j - v_{j-1}}{h} + A(u_j) \ni f_j, \quad j = \overline{1,N}, \\ v_j' + \frac{u_{j+1} - u_j}{h} + B(v_j) \ni g_j, \quad j = \overline{0,N-1}, \end{aligned} \tag{1.4}$$

with the conditions $v_0 = -\overline{\alpha}(u_0)$, $u_N = \overline{\beta}(v_N)$, $u_j(0) = u_{j0}$, $j = \overline{1,N}$, and $v_j(0) = v_{j0}$, $j = \overline{0,N-1}$.

For a unitary writing, we take $j = \overline{1,N-1}$ in both equations of the above system and then the extreme conditions become $v_0 = -\overline{\alpha}(u_1)$ and $u_N = \overline{\beta}(v_{N-1})$ (they do not show u_0 and v_N). By passing $N \to N+1$ and taking different steps h_j, we obtain the system $(\overline{S})$ in u_j, v_j, $j = \overline{1,N}$ with $v_0 = -\overline{\alpha}(u_1)$, $u_{N+1} = \overline{\beta}(v_N)$ ($\alpha = \overline{\alpha}$, $\beta = \overline{\beta}$), and $H = \mathbb{R}$.

In this way the study of the partial differential system (S) reduces to the study of the ordinary differential system $(\overline{S})$ (with $H = \mathbb{R}$ and $h_j = h$, for all j). The solution u_j, v_j depends on h and it seems that $\widetilde{u}(t,x) = \sum u_j(t)\varphi_j(x)$, $\widetilde{v}(t,x) = \sum v_j(t)\varphi_j(x)$ approximate the solution u, v of the system (S). We will not study here the convergence of the solution $\widetilde{u}$, $\widetilde{v}$ to u, v, but we will investigate the well-posedness of the discrete problem $(\overline{S}) + (\overline{EC}) + (\overline{ID})$.

We will also study the discrete system that corresponds to (S) for $x \in (0,\infty)$ with the boundary condition

$$v(t,0) = -\overline{\alpha}(u(t,0)), \quad t > 0, \tag*{$(BC)_1$}$$

and initial data

$$u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad x > 0. \tag*{$(IC)_1$}$$

More precisely we will investigate the infinite discrete hyperbolic system

$$\begin{aligned} \frac{du_n}{dt} + \frac{v_n - v_{n-1}}{h} + A(u_n) \ni f_n, \\ \frac{dv_n}{dt} + \frac{u_{n+1} - u_n}{h} + B(v_n) \ni g_n, \quad n = 1,2,\ldots,\ t \in [0,T],\ \text{in } H, \end{aligned} \tag*{$(\widetilde{S})$}$$

with the extreme condition

$$v_0(t) = -\alpha(u_1(t)), \quad t \in [0,T], \tag*{$(\widetilde{EC})$}$$

and initial data

$$u_n(0) = u_{n0}, \quad v_n(0) = v_{n0}, \quad n = 1,2,\ldots. \tag*{$(\widetilde{ID})$}$$

Although the proposed problems appeared by discretization of the problem $(S) + (BC) + (IC)$ and the corresponding one for $x \in (0,\infty)$, our problems also cover some nonlinear differential systems in Hilbert spaces.

For other classes of difference and differential equations in abstract spaces we refer the reader to [1, 2, 10].

In Section 2 we recall some definitions and results from the theory of maximal monotone operators that we need to prove our results. In Sections 3 and 4 we study the problems $(\overline{S})+(\overline{EC})+(\overline{ID})$ and $(\widetilde{S})+(\widetilde{EC})+(\widetilde{ID})$.

2. Notations and preliminaries

Let H be a real Hilbert space with the scalar product $\langle\cdot,\cdot\rangle$ and the associated norm $\|\cdot\|$. We denote by $\to$ and $\rightharpoonup$ the strong and weak convergence in H, respectively. For a multivalued operator $A: H \to H$ we denote by $D(A)=\{x\in H;\ A(x)\neq\emptyset\}$ its *domain* and by $R(A)=\cup\{A(x);\ x\in D(A)\}$ its *range*. The operator A is identified with its *graph* $G(A)=\{[x,y]\in H\times H;\ x\in D(A),\ y\in R(A)\}\subset H\times H$.

We use for A the notation $A: D(A)\subset H\to H$. If $A,B\subset H\to H$ and $\lambda\in\mathbb{R}$ then

$$\begin{gathered}\lambda A=\{[x,\lambda y];\ y\in A(x)\},\qquad D(\lambda A)=D(A),\\ A+B=\{[x,y+z];\ y\in A(x),\ z\in A(x)\},\qquad D(A+B)=D(A)\cap D(B).\end{gathered}\tag{2.1}$$

The operator $A: D(A)\subset H\to H$ is *monotone* if for all $x_1,x_2\in D(A)$ and $y_1\in A(x_1)$, $y_2\in A(x_2)$ we have $\langle y_1-y_2,x_1-x_2\rangle\geq 0$.

An operator $A: H\to H$ single-valued and everywhere defined is *hemicontinuous* if for all $x,y\in H$ we have $A(x+ty)\rightharpoonup A(x)$, as $t\to 0$. The operator $A: D(A)\subset H\to H$ is *demicontinuous* if it is strongly weakly continuous, that is, if $([x_n,y_n])_n\subset A$ with $x_n\to x$, as $n\to\infty$ and $y_n\rightharpoonup y$, as $n\to\infty$, then $[x,y]\in A$.

A demicontinuous operator is also hemicontinuous.

The operator $A: D(A)\subset H\to H$ is *maximal monotone* if it is maximal in the set of all monotone operators, that is, A is monotone and, as subset of $H\times H$, it is not properly contained in any other monotone subset of $H\times H$.

The monotone operator $A: D(A)\subset H\to H$ is maximal monotone if and only if for any $\lambda>0$ (equivalently for some $\lambda>0$), $R(I+\lambda A)=H$.

If $A: H\to H$ is everywhere defined, single-valued, monotone, and hemicontinuous, then it is maximal monotone.

If $A: D(A)\subset H\to H$ is maximal monotone and $B: H\to H$ is everywhere defined, single-valued, monotone, and hemicontinuous, then $A+B$ is maximal monotone.

For a maximal monotone operator $A: D(A)\subset H\to H$, the operators

$$J_\lambda=(I+\lambda A)^{-1}: H\longrightarrow H,\quad \lambda>0,\qquad A_\lambda=\frac{1}{\lambda}(I-J_\lambda): H\longrightarrow H,\quad \lambda>0,\tag{2.2}$$

are the *resolvent* and the *Yosida approximation* of A.

For an operator $A: D(A)\subset H\to H$, $f:(0,\infty)\to H$ and $u_0\in H$, we consider the Cauchy problem

$$\frac{du}{dt}(t)+A(u(t))\ni f(t),\quad t>0,\ u(0)=u_0.\tag{CP}$$

The function $u \in C([0,T];H)$ is *strong solution* for the problem (CP) if u is absolutely continuous on every compact of $(0,T)$, $u(t) \in D(A)$, for a.a. $t \in (0,T)$, $u(0) = u_0$, and u satisfies $(CP)_1$ for a.a. $t \in (0,T)$.

The function $u \in C([0,T];H)$ is *weak solution* for the problem (CP) if there exist $(u_n)_n \subset W^{1,\infty}(0,T;H)$ and $(f_n)_n \subset L^1(0,T;H)$ such that

$$\frac{du_n}{dt}(t) + A(u_n(t)) \ni f_n(t), \quad \text{for a.a. } t \in (0,T),\ n = 1,2,\dots, \tag{2.3}$$

$u_n \to u$, as $n \to \infty$, in $C([0,T];H)$, $u(0) = u_0$, and $f_n \to f$, as $n \to \infty$, in $L^1(0,T;H)$.

For other properties of the maximal monotone operators and for the main results of existence, uniqueness of the strong and weak solutions for the nonlinear evolution equations in Hilbert spaces, we refer the reader to [3, 6, 13].

3. The problem $(\overline{S}) + (\overline{EC}) + (\overline{ID})$

The assumptions we will use in this section are the following.

(H1) The operators $A : D(A) \subset H \to H$, $B : D(B) \subset H \to H$ are maximal monotone, possibly multivalued, with $D(A) \neq \emptyset$, $D(B) \neq \emptyset$.

(H2) The operators $\alpha, \beta : H \to H$ are single-valued and maximal monotone.

(H3) The constants $h_j > 0$, $j = \overline{1,N}$.

We will write our problem as a Cauchy problem in a certain Hilbert space, for we consider the Hilbert *space* $X = H^{2N} = \{(u_1,u_2,\dots,u_N,v_1,v_2,\dots,v_N)^T;\ u_j,v_j \in H,\ j = \overline{1,N}\}$ with the scalar product

$$\langle (u_1,\dots,u_N,v_1,\dots,v_N)^T, (\overline{u}_1,\dots,\overline{u}_N,\overline{v}_1,\dots,\overline{v}_N)^T \rangle_X = \sum_{j=1}^{N} h_j(\langle u_j,\overline{u}_j\rangle + \langle v_j,\overline{v}_j\rangle) \tag{3.1}$$

and the corresponding norm $\|\cdot\|_X$.

We introduce the *operator* $\mathcal{A} : D(\mathcal{A}) \subset X \to X$,

$$\begin{aligned}
&\mathcal{A}\big((u_1,u_2,\dots,u_N,v_1,v_2,\dots,v_N)^T\big) \\
&\quad = \left(\frac{v_1+\alpha(u_1)}{h_1}, \frac{v_2-v_1}{h_2}, \dots, \frac{v_N-v_{N-1}}{h_N}, \frac{u_2-u_1}{h_1}, \frac{u_3-u_2}{h_2}, \dots, \frac{\beta(v_N)-u_N}{h_N}\right)^T.
\end{aligned} \tag{3.2}$$

Because $D(\alpha) = D(\beta) = H$, we deduce that $D(\mathcal{A}) = X$.

We also define the *operator* $\mathcal{B} : D(\mathcal{B}) \subset X \to X$, $D(\mathcal{B}) = D(A)^N \times D(B)^N$,

$$\begin{aligned}
&\mathcal{B}((u_1,u_2,\dots,u_N,v_1,v_2,\dots,v_N)^T) \\
&\quad = \{(\gamma_1,\gamma_2,\dots,\gamma_N,\delta_1,\delta_2,\dots,\delta_N)^T;\ \gamma_i \in A(u_i),\ \delta_i \in B(v_i),\ i = \overline{1,N}\}.
\end{aligned} \tag{3.3}$$

Using the operators $\mathscr{A}$ and $\mathscr{B}$, our problem can be equivalently expressed as the following Cauchy problem in the space X

$$\frac{dU}{dt}(t)+\mathscr{A}(U(t))+\mathscr{B}(U(t))\ni F(t), \qquad U(0)=U_0, \tag{$\overline{P}$}$$

where $U=(u_1,u_2,\dots,u_N,v_1,\dots,v_N)^T$, $U_0=(u_{10},u_{20},\dots,u_{N0},v_{10},\dots,v_{N0})^T$, $F=(f_1,f_2,\dots,f_N,g_1,\dots,g_N)^T$.

Lemma 3.1. *If the assumptions (H2) and (H3) hold, then the operator $\mathscr{A}$ is monotone and demicontinuous; so it is maximal monotone.*

Proof. The operator $\mathscr{A}$ is defined on X and it is single-valued. $\mathscr{A}$ is monotone, because

$$\begin{aligned}
&\langle \mathscr{A}(U)-\mathscr{A}(\overline{U}),U-\overline{U}\rangle_X\\
&\quad=\sum_{j=1}^{N}h_j\left\langle\frac{v_j-v_{j-1}-\overline{v}_j+\overline{v}_{j-1}}{h_j},u_j-\overline{u}_j\right\rangle\\
&\qquad+\sum_{j=1}^{N}h_j\left\langle\frac{u_{j+1}-u_j-\overline{u}_{j+1}+\overline{u}_j}{h_j},v_j-\overline{v}_j\right\rangle\\
&\quad=\langle v_1+\alpha(u_1)-\overline{v}_1-\alpha(\overline{u}_1),u_1-\overline{u}_1\rangle\\
&\qquad+\sum_{j=2}^{N}\left(\langle v_j-\overline{v}_j,u_j-\overline{u}_j\rangle-\langle v_{j-1}-\overline{v}_{j-1},u_j-\overline{u}_j\rangle\right)\\
&\qquad+\sum_{j=1}^{N-1}\left(\langle u_{j+1}-\overline{u}_{j+1},v_j-\overline{v}_j\rangle-\langle u_j-\overline{u}_j,v_j-\overline{v}_j\rangle\right)\\
&\qquad+\langle\beta(v_N)-u_N-\beta(\overline{v}_N)+\overline{u}_N,v_N-\overline{v}_N\rangle\\
&\quad=\langle v_1-\overline{v}_1,u_1-\overline{u}_1\rangle+\langle\alpha(u_1)-\alpha(\overline{u}_1),u_1-\overline{u}_1\rangle\\
&\qquad-\langle u_1-\overline{u}_1,v_1-\overline{v}_1\rangle+\langle v_N-\overline{v}_N,u_N-\overline{u}_N\rangle\\
&\qquad+\langle\beta(v_N)-\beta(\overline{v}_N),v_N-\overline{v}_N\rangle-\langle u_N-\overline{u}_N,v_N-\overline{v}_N\rangle\\
&\quad=\langle\alpha(u_1)-\alpha(\overline{u}_1),u_1-\overline{u}_1\rangle+\langle\beta(v_N)-\beta(\overline{v}_N),v_N-\overline{v}_N\rangle\geq 0,
\end{aligned}\tag{3.4}$$

with $v_0=-\alpha(u_1)$, $\overline{v}_0=-\alpha(\overline{u}_1)$, $u_{N+1}=\beta(v_N)$, $\overline{u}_{N+1}=\beta(\overline{v}_{N+1})$, for all $U=(u_1,u_2,\dots,u_N,v_1,\dots,v_N)^T$, $\overline{U}=(\overline{u}_1,\overline{u}_2,\dots,\overline{u}_N,\overline{v}_1,\dots,\overline{v}_N)^T\in X$.

The operator $\mathscr{A}$ is also demicontinuous, that is, if $U^n\to U^0$ and $\mathscr{A}(U^n)\rightharpoonup V^0$, then $V^0=\mathscr{A}(U^0)$. Indeed, let $U^n=(u_1^n,u_2^n,\dots,u_N^n,v_1^n,\dots,v_N^n)$, $U^0=(u_1^0,u_2^0,\dots,u_N^0,v_1^0,\dots,v_N^0)$, $U^n\to U^0$ and $\mathscr{A}(U^n)=((v_1^n+\alpha(u_1^n))/h_1,(v_2^n-v_1^n)/h_2,\dots,(v_N^n-v_{N-1}^n)/h_N,(u_2^n-u_1^n)/h_1,\dots,(\beta(v_N^n)-u_N^n)/h_N)^T$, $V^0=(x_1^0,x_2^0,\dots,x_N^0,y_1^0,\dots,y_N^0)^T$, $\mathscr{A}(U^n)\rightharpoonup V^0$.

From $U^n \to U^0$ we deduce that

$$u_j^n \longrightarrow u_j^0, \quad v_j^n \longrightarrow v_j^0, \quad \text{for } n \longrightarrow \infty, \forall j = \overline{1,N}. \tag{3.5}$$

Because $\mathcal{A}(U^n) \rightharpoonup V^0$, we get $\langle \mathcal{A}(U^n), Y\rangle_X \to \langle V^0, Y\rangle_X$, for all $Y \in X$, $Y = (\alpha_1,\alpha_2,\dots,\alpha_N, \beta_1,\dots,\beta_N)^T$, that is,

$$\left.\begin{aligned} \langle v_1^n + \alpha(u_1^n), \alpha_1\rangle &\longrightarrow h_1\langle x_1^0, \alpha_1\rangle, \\ \langle v_j^n - v_{j-1}^n, \alpha_j\rangle &\longrightarrow h_j\langle x_j^0, \alpha_j\rangle, \quad j = \overline{2,N}, \\ \langle u_j^n - u_{j-1}^n, \beta_{j-1}\rangle &\longrightarrow h_{j-1}\langle y_{j-1}^0, \beta_{j-1}\rangle, \quad j = \overline{2,N}, \\ \langle \beta(v_N^n) - u_N^n, \beta_N\rangle &\longrightarrow h_N\langle y_N^0, \beta_N\rangle, \quad \text{in } H, \end{aligned}\right. \Longrightarrow v_1^n + \alpha(u_1^n) \rightharpoonup h_1 x_1^0, \tag{3.6}$$

$$\begin{aligned} v_j^n - v_{j-1}^n &\rightharpoonup h_j x_j^0, \quad j = \overline{2,N}, \\ u_j^n - u_{j-1}^n &\rightharpoonup h_{j-1} y_{j-1}^0, \quad j = \overline{2,N}, \end{aligned} \tag{3.7}$$

$$\beta(v_N^n) - u_N^n \rightharpoonup h_N y_N^0, \quad \text{in } H, \text{ as } n \longrightarrow \infty. \tag{3.8}$$

From the relations (3.5) and (3.7) we obtain $x_j^0 = (v_j^0 - v_{j-1}^0)/h_j$, $j = \overline{2,N}$, $y_{j-1}^0 = (u_j^0 - u_{j-1}^0)/h_{j-1}$, $j = \overline{2,N}$. Because α and β are demicontinuous, by (3.5), (3.6), and (3.8) we deduce

$$h_1 x_1^0 - v_1^0 = \alpha(u_1^0), \qquad h_N y_N^0 + u_N^0 = \beta(v_N^0) \Longrightarrow x_1^0 = \frac{v_1^0 + \alpha(u_1^0)}{h_1}, \qquad y_N^0 = \frac{\beta(v_N^0) - u_N^0}{h_N}. \tag{3.9}$$

Therefore $V^0 = \mathcal{A}(U^0)$. Hence the operator $\mathcal{A}$ is demicontinuous (so it is also hemicontinuous) and, by [6, Proposition 2.4] we deduce that it is maximal monotone. □

Lemma 3.2. *If the assumptions (H1) and (H3) hold, then the operator $\mathcal{B}$ is maximal monotone in X.*

Proof. The operator $\mathcal{B}$ is evidently monotone

$$\langle Z - \overline{Z}, U - \overline{U}\rangle_X = \sum_{j=1}^{N} h_j\left(\langle \gamma_j - \overline{\gamma}_j, u_j - \overline{u}_j\rangle + \langle \delta_j - \overline{\delta}_j, v_j - \overline{v}_j\rangle\right) \geq 0, \tag{3.10}$$

for all $U = (u_1,u_2,\dots,u_N,v_1,\dots,v_N)^T$, $\overline{U} = (\overline{u}_1,\overline{u}_2,\dots,\overline{u}_N,\overline{v}_1,\dots,\overline{v}_N)^T \in D(\mathcal{B})$, $Z \in \mathcal{B}(U)$, $\overline{Z} \in \mathcal{B}(\overline{U})$, for all $\gamma_j \in A(u_j)$, $\overline{\gamma}_j \in A(\overline{u}_j)$, $\delta_j \in B(v_j)$, $\overline{\delta}_j \in B(\overline{v}_j)$, $j = \overline{1,N}$.

It is also maximal monotone in X. Indeed, by [6, Proposition 2.2] it is sufficient (and necessary) to show that for $\lambda > 0$, $R(I + \lambda\mathcal{B}) = X \Leftrightarrow$ for all $Y \in X$, $Y = (x_1,x_2,\dots,x_N,y_1,\dots,y_N)^T$ there exists $U \in X$, $U = (u_1,u_2,\dots,u_N,v_1,\dots,v_N)^T$ such that

$$U + \lambda\mathcal{B}(U) \ni Y. \tag{3.11}$$

The last relation gives us

$$\begin{aligned} u_j+\lambda\gamma_j &= x_j, \quad j=\overline{1,N}, \\ v_j+\lambda\delta_j &= y_j, \quad j=\overline{1,N}, \end{aligned} \Longrightarrow \begin{aligned} u_j &= (I+\lambda A)^{-1}(x_j)=J_\lambda^A(x_j), \quad j=\overline{1,N}, \\ v_j &= (I+\lambda B)^{-1}(y_j)=J_\lambda^B(y_j), \quad j=\overline{1,N}, \end{aligned} \tag{3.12}$$

where $\gamma_j \in A(u_j)$, $\delta_j \in B(v_j)$ $j=\overline{1,N}$, and J_λ^A and J_λ^B are the resolvents of A and B, respectively (A and B are maximal monotone). Then $U=(u_1,\dots,u_N,v_1,\dots,v_N)^T$, where u_j and v_j, $j=\overline{1,N}$, defined above, satisfy our condition (3.11). □

We give now the main result for our initial problem $(\overline{S})+(\overline{EC})+(\overline{ID})$.

Theorem 3.3. *Assume that the assumptions (H1)–(H3) hold. If $u_{j0}\in D(A)$, for all $j=\overline{1,N}$, $v_{j0}\in D(B)$, for all $j=\overline{1,N}$, and $f_j,g_j\in W^{1,1}(0,T;H)$, $j=\overline{1,N}$, then there exist unique functions u_j and $v_j\in W^{1,\infty}(0,T;H)$, $j=\overline{1,N}$, $u_j(t)\in D(A)$, $v_j(t)\in D(B)$, for all $j=\overline{1,N}$, for all $t\in[0,T]$, which verify the system $(\overline{S})$ for every $t\in[0,T)$, the condition $(\overline{EC})$ for every $t\in[0,T)$, and the initial data $(\overline{ID})$.*

Moreover u_j and v_j, $j=\overline{1,N}$, are everywhere differentiable from right in the topology of H and

$$\begin{aligned} \frac{d^+u_j}{dt} &= \left(f_j-A(u_j)-\frac{v_j-v_{j-1}}{h_j}\right)^0, \quad j=\overline{1,N}, \\ \frac{d^+v_j}{dt} &= \left(g_j-B(v_j)-\frac{u_{j+1}-u_j}{h_j}\right)^0, \quad j=\overline{1,N},\ \forall t\in[0,T), \end{aligned} \tag{3.13}$$

with $v_0(t)=-\alpha(u_1(t))$, $u_{N+1}(t)=\beta(v_N(t))$, $\forall t\in[0,T)$.

Proof. Because the operator $\mathcal{B}$ is maximal monotone in X and $\mathcal{A}$ is single-valued, with $D(\mathcal{A})=X$, monotone, and hemicontinuous, by [3, Corollary 1.3, Chapter II] we deduce that $\mathcal{A}+\mathcal{B}:D(\mathcal{B})\subset X\to X$ is maximal monotone. By [3, Theorem 2.2, Corollary 2.1, Chapter III] we deduce that, for $U_0\in D(\mathcal{B})$ and $F\in W^{1,1}(0,T;X)$, the problem $(\overline{P})$ has a unique solution $U=(u_1,u_2,\dots,u_N,v_1,\dots,v_N)^T\in W^{1,\infty}(0,T;X)$, $U(t)\in D(\mathcal{B})$, for all $t\in[0,T)$. We consider $(\overline{P})_1$ in the interval $[0,T+\varepsilon]$, $\varepsilon>0$, (by extending correspondingly the functions f_j and g_j, $j=\overline{1,N}$) and we get $U(T)\in D(\mathcal{B})$.

The solution U is everywhere differentiable from right and

$$\frac{d^+U}{dt}(t)=\left(F(t)-\mathcal{A}(U(t))-\mathcal{B}(U(t))\right)^0, \quad \forall t\in[0,T), \tag{3.14}$$

that is, the relations from theorem are verified. In addition we have

$$\left\|\frac{d^+U}{dt}(t)\right\|_X \le \left\|\left(F(0)-\mathcal{A}(U_0)-\mathcal{B}(U_0)\right)^0\right\|_X+\int_0^t\left\|\frac{dF}{ds}(s)\right\|_X ds, \quad \forall t\in[0,T). \tag{3.15}$$

If U and V are the solutions of $(\overline{P})$ corresponding to (U_0,F), $(V_0,G) \in D(\mathcal{B}) \times W^{1,1}(0,T;X)$, then

$$\|U(t) - V(t)\|_X \le \|U_0 - V_0\|_X + \int_0^t \|F(s) - G(s)\|_X ds, \quad \forall t \in [0,T]. \tag{3.16}$$

□

Remark 3.4. If $U_0 \in \overline{D(\mathcal{B})} = \overline{D(A)}^N \times \overline{D(B)}^N$ and $F \in L^1(0,T;X)$, then, by [3, Corollary 2.2, Chapter III], the problem $(\overline{P}) \Leftrightarrow (\overline{S}) + (\overline{EC}) + (\overline{ID})$ has a unique weak solution $U \in C([0,T];X)$, that is, there exist $(F_n)_n \subset W^{1,1}(0,T;X)$, $F_n \to F$, as $n \to \infty$, in $L^1(0,T;X)$ and $(U_n)_n \subset W^{1,\infty}(0,T;X)$, $U_n(0) = U_0$, $U_n \to U$, as $n \to \infty$ in $C([0,T];X)$, strong solutions for the problems

$$\frac{dU_n}{dt}(t) + (\mathcal{A} + \mathcal{B})(U_n(t)) \ni F_n(t), \quad \text{for a.a. } t \in (0,T),\ n = 1,2,\dots. \tag{3.17}$$

4. The problem $(\widetilde{S}) + (\widetilde{EC}) + (\widetilde{ID})$

We present the assumptions that we will use in this section as follows.

$(\widetilde{\mathrm{H}}1)$ The operators $A : D(A) \subset H \to H$ and $B : D(B) \subset H \to H$ are maximal monotone, $0 \in A(0)$, $0 \in B(0)$, and there exist $a_1, a_2 > 0$ such that

$$\|\gamma\| \le a_1 \|u\|, \quad \forall u \in D(A),\ \forall \gamma \in A(u); \qquad \|\delta\| \le a_2 \|u\|, \quad \forall u \in D(B),\ \forall \delta \in B(u). \tag{4.1}$$

$(\widetilde{\mathrm{H}}2)$ The operator $\alpha : H \to H$ is single-valued and maximal monotone.

$(\widetilde{\mathrm{H}}3)$ The constant $h > 0$.

We consider the *space* $Y = l_h^2(H) \times l_h^2(H)$, where $l_h^2(H) = \{(u_n)_n \subset H; \sum_{n=1}^{\infty} \|u_n\|^2 < \infty\}$ $(= l^2(H))$, with the scalar product

$$\begin{aligned}\langle ((u_n)_n, (v_n)_n), ((\overline{u}_n)_n, (\overline{v}_n)_n) \rangle_Y &= \langle (u_n)_n, (\overline{u}_n)_n \rangle_{l_h^2(H)} + \langle (v_n)_n, (\overline{v}_n)_n \rangle_{l_h^2(H)} \\ &= \sum_{n=1}^{\infty} h \langle u_n, \overline{u}_n \rangle + \sum_{n=1}^{\infty} h \langle v_n, \overline{v}_n \rangle.\end{aligned} \tag{4.2}$$

We define the *operator* $\widetilde{\mathcal{A}} : Y \to Y$,

$$\widetilde{\mathcal{A}}((u_n)_n, (v_n)_n) = \left(\left(\frac{v_n - v_{n-1}}{h} \right)_n, \left(\frac{u_{n+1} - u_n}{h} \right)_n \right), \quad \text{with } v_0 = -\alpha(u_1), \tag{4.3}$$

and the *operator* $\widetilde{\mathcal{B}} : D(\widetilde{\mathcal{B}}) \subset Y \to Y$,

$$\widetilde{\mathcal{B}}((u_n)_n, (v_n)_n) = \{((\gamma_n)_n, (\delta_n)_n) \in Y,\ \gamma_n \in A(u_n),\ \delta_n \in B(v_n),\ \forall n \ge 1\}, \tag{4.4}$$

with $D(\widetilde{\mathcal{B}}) = \{((u_n)_n, (v_n)_n) \in Y;\ u_n \in D(A),\ v_n \in D(B),\ \forall n \ge 1\}$.

Lemma 4.1. *If the assumptions $(\widetilde{H2})$ and $(\widetilde{H3})$ hold, then the operator $\widetilde{\mathcal{A}}$ is monotone and demicontinuous in Y.*

Proof. First we observe that $\widetilde{\mathcal{A}}$ is well-defined in Y. If $((u_n)_n,(v_n)_n) \in Y$, then $\widetilde{\mathcal{A}}((u_n)_n,(v_n)_n) \in Y$, and $D(\widetilde{\mathcal{A}}) = Y$.

The operator $\widetilde{\mathcal{A}}$ is monotone, because

$$
\begin{aligned}
&\langle \widetilde{\mathcal{A}}((u_n)_n,(v_n)_n) - \widetilde{\mathcal{A}}((\overline{u}_n)_n,(\overline{v}_n)_n),((u_n)_n,(v_n)_n) - ((\overline{u}_n)_n,(\overline{v}_n)_n)\rangle_Y \\
&\quad = \left\langle \left(\frac{v_n - v_{n-1}}{h}\right)_n - \left(\frac{\overline{v}_n - \overline{v}_{n-1}}{h}\right)_n, (u_n - \overline{u}_n)_n \right\rangle_{l_h^2(H)} \\
&\qquad + \left\langle \left(\frac{u_{n+1} - u_n}{h}\right)_n - \left(\frac{\overline{u}_{n+1} - \overline{u}_n}{h}\right)_n, (v_n - \overline{v}_n)_n \right\rangle_{l_h^2(H)} \\
&\quad = \langle \alpha(u_1) - \alpha(\overline{u}_1), u_1 - \overline{u}_1\rangle \geq 0.
\end{aligned}
\tag{4.5}
$$

Next we prove that $\widetilde{\mathcal{A}}$ is demicontinuous, that is, if

$$
((u_n^j)_n,(v_n^j)_n) \longrightarrow ((u_n^0)_n,(v_n^0)_n), \quad \text{for } j \longrightarrow \infty \text{ in } Y, \tag{4.6}
$$

$$
\widetilde{\mathcal{A}}((u_n^j)_n,(v_n^j)_n) \rightharpoonup ((x_n)_n,(y_n)_n), \quad \text{for } j \longrightarrow \infty \text{ in } Y, \tag{4.7}
$$

then $((x_n)_n,(y_n)_n) = \widetilde{\mathcal{A}}((u_n^0)_n,(v_n^0)_n)$.

From (4.6) we deduce

$$
\begin{gathered}
\sqrt{h\|(u_n^j)_n - (u_n^0)_n\|_{l^2(H)}^2 + h\|(v_n^j)_n - (v_n^0)_n\|_{l^2(H)}^2} \longrightarrow 0, \quad \text{for } j \longrightarrow \infty \Longrightarrow \\
\begin{matrix} \|(u_n^j)_n - (u_n^0)_n\|_{l^2(H)} \longrightarrow 0, & \text{for } j \longrightarrow \infty \\ \|(v_n^j)_n - (v_n^0)_n\|_{l^2(H)} \longrightarrow 0, & \text{for } j \longrightarrow \infty \end{matrix} \Longrightarrow \\
\begin{matrix} \displaystyle\sum_{n=1}^{\infty} \|u_n^j - u_n^0\|^2 \longrightarrow 0, & \text{for } j \longrightarrow \infty \\ \displaystyle\sum_{n=1}^{\infty} \|v_n^j - v_n^0\|^2 \longrightarrow 0, & \text{for } j \longrightarrow \infty \end{matrix} \Longrightarrow \begin{matrix} u_n^j \longrightarrow u_n^0, & \text{for } j \longrightarrow \infty, \ \forall n, \\ v_n^j \longrightarrow v_n^0, & \text{for } j \longrightarrow \infty, \ \forall n. \end{matrix}
\end{gathered}
\tag{4.8}
$$

Then by (4.7) we have

$$
\begin{aligned}
&\langle \widetilde{\mathcal{A}}((u_n^j)_n,(v_n^j)_n),((\alpha_n)_n,(\beta_n)_n)\rangle_Y \\
&\quad \longrightarrow \langle ((x_n)_n,(y_n)_n),((\alpha_n)_n,(\beta_n)_n)\rangle_Y, \quad \text{as } j \longrightarrow \infty, \ \forall ((\alpha_n)_n,(\beta_n)_n) \in Y, \\
&\quad \Longrightarrow \left\langle \left(\left(\frac{v_n^j - v_{n-1}^j}{h}\right)_n, \left(\frac{u_{n+1}^j - u_n^j}{h}\right)_n\right), ((\alpha_n)_n,(\beta_n)_n) \right\rangle_Y,
\end{aligned}
$$

$$
\begin{aligned}
&\longrightarrow \sum_{n=1}^{\infty} h\langle x_n,\alpha_n\rangle + \sum_{n=1}^{\infty} h\langle y_n,\beta_n\rangle, \quad \text{as } j \longrightarrow \infty, \text{ with } v_0^j = -\alpha(u_1^j),\ \forall\, j \geq 1, \\
&\Longrightarrow \sum_{n=1}^{\infty} \langle v_n^j - v_{n-1}^j,\alpha_n\rangle + \sum_{n=1}^{\infty} \langle u_{n+1}^j - u_n^j,\beta_n\rangle \\
&\longrightarrow \sum_{n=1}^{\infty} h\langle x_n,\alpha_n\rangle + \sum_{n=1}^{\infty} h\langle y_n,\beta_n\rangle, \quad \text{as } j \longrightarrow \infty.
\end{aligned} \tag{4.9}
$$

We take $\alpha_n = 0$, for all $n \geq 1$, and $(\beta_n)_n = (x,0,0,\ldots), (0,x,0,\ldots),\ldots (x \in H)$; we obtain by (4.8) $y_n = (u_{n+1}^0 - u_n^0)/h$, for all $n \geq 1$. For $\beta_n = 0$, for all $n \geq 1$ and $(\alpha_n)_n = (0,x,0,\ldots)$, $(0,0,x,\ldots)$, …, $(x \in H)$ we obtain by (4.8) that $x_n = (v_n^0 - v_{n-1}^0)/h$, $n \geq 2$. For $\beta_n = 0$, for all $n \geq 1$ and $\alpha_n = (x,0,0,\ldots)$ we find $(v_1^j + \alpha(u_1^j))/h \rightharpoonup x_1$, so

$$
\alpha(u_1^j) = v_1^j + \alpha(u_1^j) - v_1^j \longrightarrow hx_1 - v_1^0, \quad \text{as } j \longrightarrow \infty \tag{4.10}
$$

($v_1^j \to v_1^0$ as $j \to \infty$, by (4.8) with $n = 1$). Now since α is a demicontinuous operator, and moreover $u_1^j \to u_1^0$, as $j \to \infty$ (by (4.8) with $n = 1$), we have

$$
hx_1 - v_1^0 = \alpha(u_1^0) \Longleftrightarrow x_1 = \frac{v_1^0 + \alpha(u_1^0)}{h}. \tag{4.11}
$$

Therefore $(((v_n^0 - v_{n-1}^0)/h)_n, ((u_{n+1}^0 - u_n^0)/h)_n) = ((x_n)_n, (y_n)_n)$, $(v_0^0 = -\alpha(u_1^0))$. We deduce that $\widetilde{\mathcal{A}}$ is demicontinuous, so it is maximal monotone. □

Lemma 4.2. *If the assumptions $(\widetilde{H1})$ and $(\widetilde{H3})$ hold, then the operator $\widetilde{\mathcal{B}}$ is maximal monotone in Y.*

Proof. We suppose without loss of generality (for an easy writing) that A and B are single-valued. The operator $\widetilde{\mathcal{B}}$ under the assumptions of this lemma is well-defined in Y. Indeed, for $((u_n)_n,(v_n)_n) \in D(\widetilde{\mathcal{B}}) \Leftrightarrow ((u_n)_n,(v_n)_n) \in Y$, $u_n \in D(A)$, $v_n \in D(B)$, for all $n \geq 1$, we have $\widetilde{\mathcal{B}}((u_n)_n,(v_n)_n) \in Y$, that is, $(A(u_n))_n, (B(v_n))_n \in l^2(H)$.

By $(\widetilde{\mathrm{H}1})$ we have

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \|A(u_n)\|^2 \leq \sum_{n=1}^{\infty} a_1^2 \|u_n\|^2 = a_1^2 \|(u_n)_n\|_{l^2(H)}^2 < \infty, \\
&\sum_{n=1}^{\infty} \|B(v_n)\|^2 \leq \sum_{n=1}^{\infty} a_2^2 \|v_n\|^2 = a_2^2 \|(v_n)_n\|_{l^2(H)}^2 < \infty.
\end{aligned} \tag{4.12}
$$

The operator $\widetilde{\mathscr{B}}$ is monotone

$$\begin{aligned}
&\langle\widetilde{\mathscr{B}}((u_n)_n,(v_n)_n)-\widetilde{\mathscr{B}}((\overline{u}_n)_n,(\overline{v}_n)_n),((u_n)_n,(v_n)_n)-((\overline{u}_n)_n,(\overline{v}_n)_n)\rangle_Y\\
&\quad=\langle((A(u_n))_n,(B(v_n))_n)-((A(\overline{u}_n))_n,(B(\overline{v}_n))_n),((u_n-\overline{u}_n)_n,(v_n-\overline{v}_n)_n)\rangle_Y\\
&\quad=\langle(A(u_n)-A(\overline{u}_n))_n,(u_n-\overline{u}_n)_n\rangle_{l_h^2(H)}+\langle(B(v_n)-B(\overline{v}_n))_n,(v_n-\overline{v}_n)_n\rangle_{l_h^2(H)}\\
&\quad=\sum_{n=1}^{\infty}h\langle A(u_n)-A(\overline{u}_n),u_n-\overline{u}_n\rangle+\sum_{n=1}^{\infty}h\langle B(v_n)-B(\overline{v}_n),v_n-\overline{v}_n\rangle\geq 0,\\
&\qquad\qquad\forall((u_n)_n,(v_n)_n),((\overline{u}_n)_n,(\overline{v}_n)_n)\in D(\widetilde{\mathscr{B}}).
\end{aligned}\tag{4.13}$$

Moreover $\widetilde{\mathscr{B}}$ is maximal monotone, that is,

$$\forall\lambda>0,\quad R(I+\lambda\widetilde{B})=Y\Longleftrightarrow\forall Z=((x_n)_n,(y_n)_n)\in Y,\quad\exists W=((u_n)_n,(v_n)_n)\in D(\widetilde{\mathscr{B}})\tag{4.14}$$

such that $W+\lambda\widetilde{\mathscr{B}}(W)=Z$. The last relation is equivalent to

$$\begin{gathered}
((u_n)_n,(v_n)_n)+\lambda((A(u_n))_n,(B(v_n))_n)=((x_n)_n,(y_n)_n)\Longleftrightarrow\\
\begin{matrix}(u_n)_n+\lambda(A(u_n))_n=(x_n)_n\\(v_n)_n+\lambda(B(v_n))_n=(y_n)_n\end{matrix}\Longrightarrow\begin{matrix}u_n+\lambda A(u_n)=x_n,\\v_n+\lambda B(v_n)=y_n,\quad\forall n\geq 1\end{matrix}\Longrightarrow\\
u_n=(I+\lambda A)^{-1}(x_n)=J_\lambda^A(x_n),\quad v_n=(I+\lambda B)^{-1}(y_n)=J_\lambda^B(y_n),\quad\forall n\geq 1.
\end{gathered}\tag{4.15}$$

Because $A(0)=0$ we have $J_\mu^A(0)=0$, for all $\mu>0$ and

$$||J_\mu^A(x)-J_\mu^A(0)||\leq\|x\|\Longrightarrow||J_\mu^A(x)||\leq\|x\|,\quad\forall x\in H,\ \forall\mu>0.\tag{4.16}$$

Similarly by $B(0)=0$ we deduce $J_\mu^B(0)=0$, for all $\mu>0$, and $\|J_\mu^B x\|\leq\|x\|$, for all $x\in H$, for all $\mu>0$.

With this remark we have

$$\sum_{n=1}^{\infty}||J_\lambda^A(x_n)||^2\leq\sum_{n=1}^{\infty}||x_n||^2<\infty,\qquad\sum_{n=1}^{\infty}||J_\lambda^B(y_n)||^2\leq\sum_{n=1}^{\infty}||y_n||^2<\infty,\tag{4.17}$$

so $W=((u_n)_n,(v_n)_n)\in D(\widetilde{\mathscr{B}})$.

Using the operators $\widetilde{\mathscr{A}}$ and $\widetilde{\mathscr{B}}$, the problem $(\widetilde{S})+(\widetilde{EC})+(\widetilde{ID})$ can be written as

$$\begin{aligned}
&\frac{dV}{dt}(t)+\widetilde{\mathscr{A}}(V(t))+\widetilde{\mathscr{B}}(V(t))\ni\widetilde{F}(t),\quad t\in(0,T),\text{ in }Y\\
&V(0)=V_0,
\end{aligned}\tag{$\widetilde{P}$}$$

where $V=((u_n)_n,(v_n)_n)$, $V_0=((u_{n0})_n,(v_{n0})_n)$, and $\widetilde{F}=((f_n)_n,(g_n)_n)$. □

Theorem 4.3. *Assume that the assumptions* $(\widetilde{H1})$–$(\widetilde{H3})$ *hold. If* $V_0 \in D(\widetilde{\mathscr{B}})$ $(u_{n0} \in D(A)$, $v_{n0} \in D(B)$, *for all* $n \geq 1$ *with* $(u_{n0})_n, (v_{n0})_n \in l^2(H))$ *and* $\widetilde{F} \in W^{1,1}(0,T;Y)$ $((f_n)_n, (g_n)_n \in W^{1,1}(0,T;l^2(H)))$, *then there exists a unique function* $V = ((u_n)_n, (v_n)_n) \in W^{1,\infty}(0,T;Y)$, *with* $V(t) \in D(\widetilde{B})$, *for all* $t \in [0,T]$, *and* V *verifies the system* $(\widetilde{S})$, *for all* $t \in [0,T)$, *the extreme condition* $(\widetilde{EC})$, *for all* $t \in [0,T)$, *and the initial data* $(\widetilde{ID})$.

Proof. By Lemmas 4.1 and 4.2 we have $D(\widetilde{\mathscr{A}} + \widetilde{\mathscr{B}}) = D(\widetilde{\mathscr{B}})$ and, by [3, Corollary 1.3, Chapter II], $\widetilde{\mathscr{A}} + \widetilde{\mathscr{B}}$ is maximal monotone. Using again [3, Theorem 2.2, Chapter III] we obtain the conclusion of the theorem. In addition u_n and v_n are everywhere differentiable from the right on $[0,T)$ and, by extended f_n and g_n on $[0,T+\varepsilon]$ with $\varepsilon > 0$, we have $V(t) \in D(\widetilde{\mathscr{B}})$, for all $t \in [0,T]$. □

Remark 4.4. Although the operators $\mathscr{A}$ and $\widetilde{\mathscr{A}}$ have semblable forms, we cannot establish connections between them, because the spaces, where these operators are defined, are different. Indeed, defining $u_n = v_n = 0$, for all $n \geq N+1$, we have

$$\widetilde{\mathscr{A}}((u_n)_n, (v_n)_n) = \left(\frac{v_1 - v_0}{h}, \dots, \frac{v_N - v_{N-1}}{h}, -\frac{v_N}{h}, \frac{u_2 - u_1}{h}, \dots, \frac{u_N - u_{N-1}}{h}, -\frac{u_N}{h}\right). \tag{4.18}$$

Therefore $\widetilde{\mathscr{A}}((u_n)_n, (v_n)_n) \in H^{N+1} \times H^N \neq X$. Thus, Theorem 3.3 is not a consequence of Theorem 4.3.

Acknowledgment

The author would like to thank the referees for the careful reading of the manuscript and for their comments which led to this revised version.

References

[1] R. P. Agarwal and D. O'Regan, *Difference equations in abstract spaces*, Journal of Australian Mathematical Society. Series A **64** (1998), no. 2, 277–284.
[2] R. P. Agarwal, D. O'Regan, and V. Lakshmikantham, *Discrete second order inclusions*, Journal of Difference Equations and Applications **9** (2003), no. 10, 879–885.
[3] V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Noordhoff, Leiden, 1976.
[4] ______, *Nonlinear boundary-value problems for a class of hyperbolic systems*, Revue Roumaine de Mathématiques Pures et Appliquées **22** (1977), no. 2, 155–168.
[5] V. Barbu and G. Moroşanu, *Existence for a nonlinear hyperbolic system*, Nonlinear Analysis **5** (1981), no. 4, 341–353.
[6] H. Brézis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, North-Holland, Amsterdam, 1973.
[7] K. L. Cooke and D. W. Krumme, *Differential-difference equations and nonlinear initial-boundary value problems for linear hyperbolic partial differential equations*, Journal of Mathematical Analysis and Applications **24** (1968), no. 2, 372–387.
[8] A. Halanay and V. Răsvan, *Frequency domain conditions for forced oscillations in difference systems*, Revue Roumaine des Sciences Techniques. Série Électrotechnique et Énergétique **24** (1979), no. 1, 141–148.

[9] V. Iftimie, *Sur un problème non linéaire pour un système hyperbolique*, Revue Roumaine de Mathématiques Pures et Appliquées **21** (1976), no. 3, 303–328.
[10] E. Liz and M. Pituk, *Asymptotic estimates and exponential stability for higher-order monotone difference equations*, Advances in Difference Equations **2005** (2005), no. 1, 41–55.
[11] R. Luca, *An existence result for a nonlinear hyperbolic system*, Differential and Integral Equations **8** (1995), no. 4, 887–900.
[12] G. Moroşanu, *Mixed problems for a class of nonlinear differential hyperbolic systems*, Journal of Mathematical Analysis and Applications **83** (1981), no. 2, 470–485.
[13] ______, *Nonlinear Evolution Equations and Applications*, Mathematics and Its Applications (East European Series), vol. 26, D. Reidel, Dordrecht; Editura Academiei, Bucharest, 1988.
[14] A. Rousseau, R. Temam, and J. Tribbia, *Boundary conditions for an ocean related system with a small parameter*, Nonlinear Partial Differential Equations and Related Analysis, Contemporary Mathematics, vol. 371, American Mathematical Society, Rhode Island, 2005, pp. 231–263.

Rodica Luca: Department of Mathematics, "Gheorghe Asachi" Technical University of Iaşi, Boulevard Carol I number 11, Iaşi 700506, Romania
E-mail address: rluca@math.tuiasi.ro